Laurent polynomials and deformations of Gorenstein toric singularities

Motivation: We want to classify Fano manifolds using mirror symmetry.

$$\left\{ \begin{array}{l} \text{Fano manifolds} \\ \text{with mild singularities} \end{array} \right\} \Big/ \begin{array}{l}\mathbb{Q}\text{-Gorenstein} \\ \text{deformation}\end{array} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{Laurent polynomials} \\ \text{with nice properties} \end{array} \right\} \Big/ \text{mutations}$$

- Gross–Siebert program and its generalizations (i.e. Felten–F–Ruddat, Ruddat–Corti–Grafnitz)
- Laurent inversion (Coates – Prince – Kasprzyk)

P lattice polytope $\subset N_{\mathbb{R}}$, $\sigma := \operatorname{cone}(P) \subseteq \widetilde{N}_{\mathbb{R}} := (N \oplus \mathbb{Z})_{\mathbb{R}}$,

$\underline{X_P := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \widetilde{M}]}$ affine Gorenstein toric variety.

$f \in \mathbb{C}[N]$ Laurent polynomial

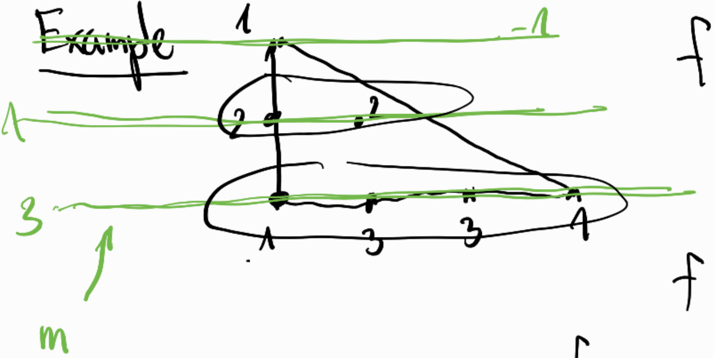
<u>Def</u> f is (m, g)-mutable if we can write $f = \sum_{i \in \mathbb{Z}} f_i$, where $f_i \in \mathbb{C}[(m = i) \cap N]$ such that $\frac{f_i}{g^i}$ is a Laurent polynomial.

$m \in \widetilde{M}$ (affine function on N)

We denote $\operatorname{mut}_m^g f := \sum_i \frac{f_i}{g^i}$.

$f_3 \qquad f_1 \qquad f_{-1}$

Example

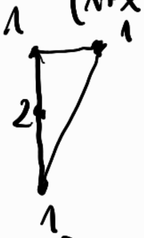


$f = (1+3x+3x^2+x^3) + (2y+2xy) + (y)$

$g = 1+x$

f is (m,g)-mutable since

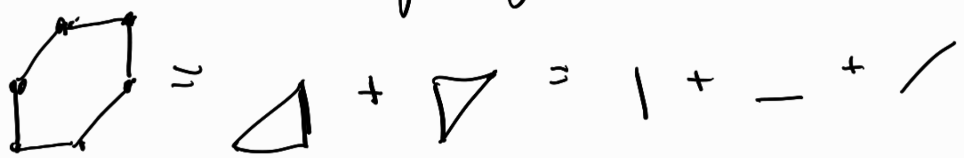
$$\frac{f_3}{(1+x)^3} = 1, \quad \frac{f_1}{1+x} = \frac{2y(1+x)}{1+x} = 2y$$

$\boxed{\mathrm{mut}_m^g f} =$  $= \boxed{1+2y+y^2+y^2x}$

Deformation theory of affine toric varieties

- 88 Kollár–Shepherd-Barron ⇝ construct miniversal def. of 2-dim affine toric varieties.
- 97 Altmann ⇝ constr. miniversal deformation of X_P with only isolated singularities.

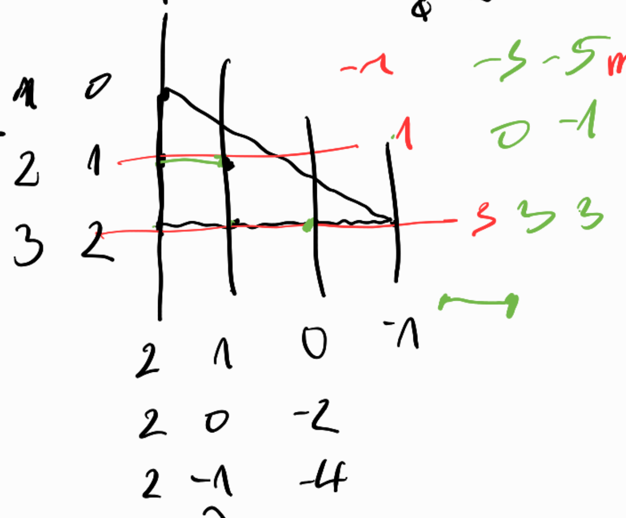
If P is a lattice polygon, then X_P isolated $\Leftrightarrow$ every edge of P has lattice length 1.



$$T^1_X = T^1_X(-R^*)$$

If X_P is not isolated, then $\dim_{\mathbb{C}} T^1_X = \infty$

Example



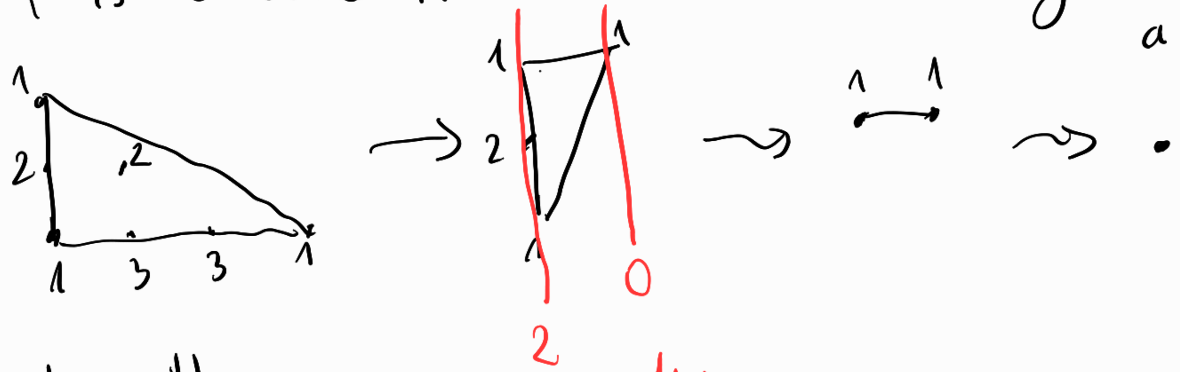
$-m \in \tilde{\Pi}$

$\dim_{\mathbb{C}} T^1_X(-m) = 1$

Conjecture (Corti - F - Petracci)

$$\left\{ \begin{array}{l} \text{Smoothing minimal} \\ \text{components of } X_P \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{0-mutable} \\ \text{Laurent polynomials } f \text{ with } \Delta(f) = P \end{array} \right\}$$

Def f is 0-mutable if we can mutate it to g with $\Delta(g)$ is a point



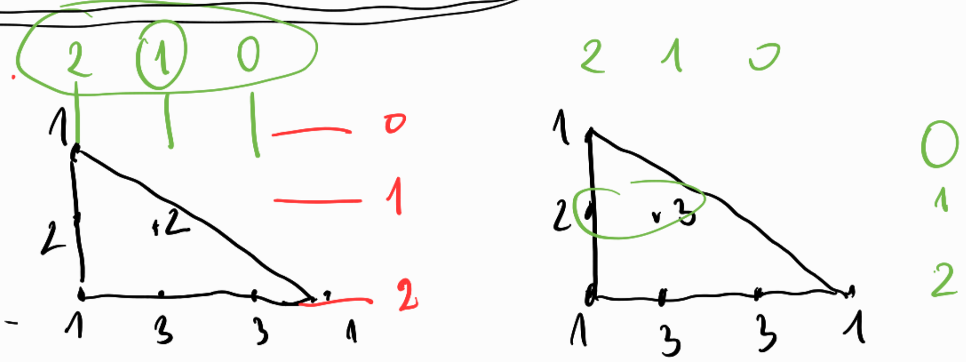
f in two-variable

Definition f is m-mutable if it is (m, g)-mutable with $\Delta(g)$ a line segment of lattice length 1. ($g = 1+y$)

Proposition (F25) If f is m-mutable ($\tilde{m}$), then we can explicitly construct 1-parameter deformation of $X_{\Delta(f)}$ over $\mathbb{C}[t_m]$ using equational flatness criterion (Altmann - F - Constantinescu)

Theorem (F25) $X_{\Delta(f)}$ is unobstructed in $\{t_m \mid f \text{ is m-mutable}\}$, meaning that there exist a formal def. family over $\mathbb{C}[[t_m \mid f \text{ is m-mutable}]]$ with injective Kodaira-Spencer map.

Example



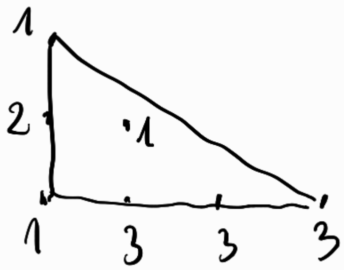
Conjecture (F25)

$\left\{ \begin{array}{l} \text{deformation components} \\ \text{of } X_P \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{maximally-mutable} \\ \text{Laurent polynomials } f \text{ with} \\ \Delta(f) = P \end{array} \right\}$

<u>Def</u> f is maximally mutable if $\nexists\, g$ such that

$$\mathcal{M}(g) \supsetneq \mathcal{M}(f) = \{m \,;\, f \text{ is } m\text{-mutable}\}$$

$$\mathcal{M}(g) = \{m \,;\, g \text{ is } m\text{-mutable}\}$$



Idea of the proof of the theorem:

<u>Th 1</u> $X_{\Delta(f)}$ is unobstructed in $\underline{\{t_m \,;\, m \in \mathcal{M}(f)\}}$

$\iff X_{\Delta(\mathrm{mut}_m^g f)}$ is unobstructed in $\{t_r \,;\, r \in \mathrm{mut}_m^g f\}$

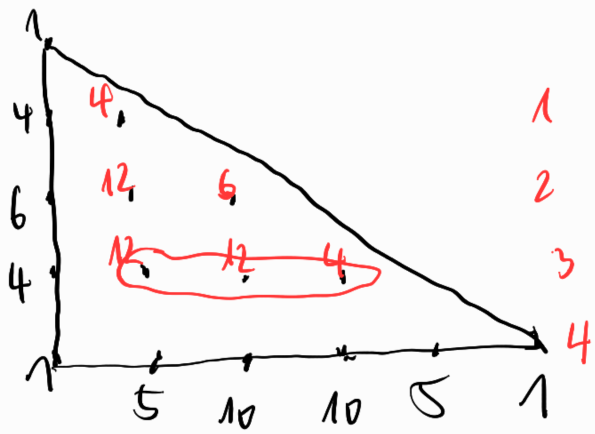
<u>Th 2</u> f can be mutated to a Laurent polynomial g such that $\exists\, v \in \underline{\Delta(g)} \cap N$ with $\underline{m(v) \leq 0}$ for every $m \in \mathcal{M}(g)$.

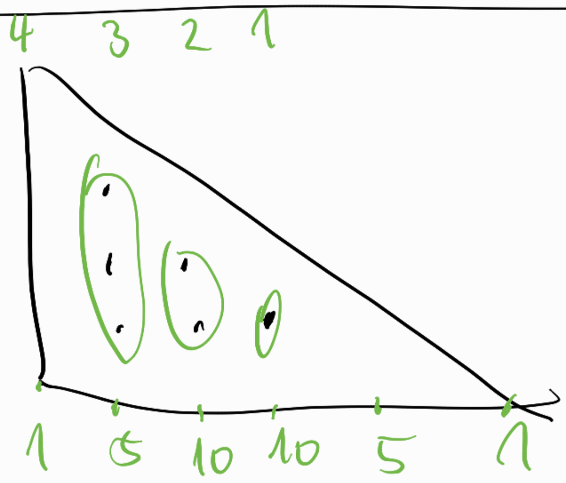
It is known that $X_{\Delta(g)}$ is unobstructed in

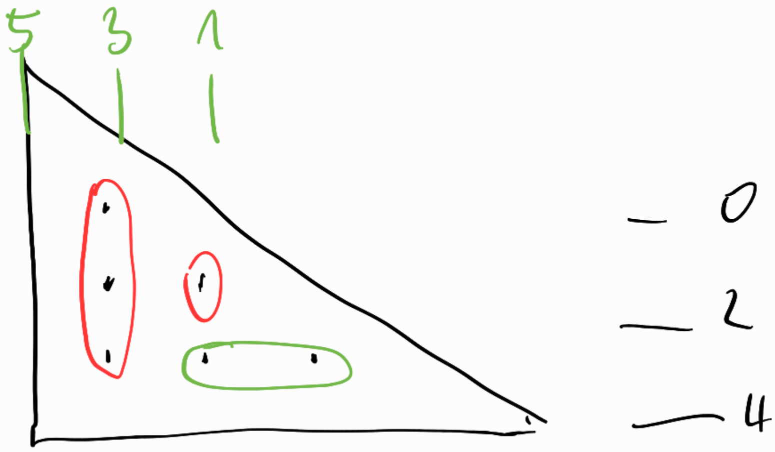
$\{t_m \mid g$ is m-mutable$\}$

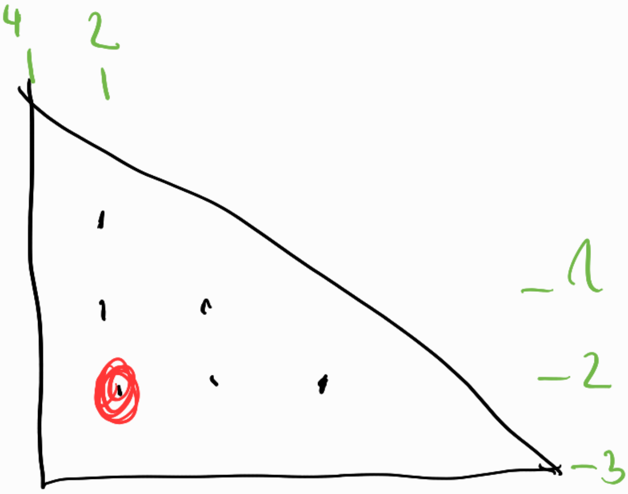
$\{ t_m \mid g \text{ is in [illegible]} \}$

Example









$(1, n, n+1)$ $n \geq 3$ 3 smooth components

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